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ON FRANKL'S PROBLEM FOR A MIXED-TYPE EQUATION WITH A SINGULAR COEFFICIENT

Boltaev Kh.Kh. O'rinboeva R.Q.
National Pedagogical University of Uzbekistan, Tashkent, Uzbekistan
Tashkent International University, Tashkent, Uzbekistan
bkhabibzhan2020@mail.ru, qodirovnarano@gmail.com

ANATATION

The paper considers a problem with the Frankl condition on different parts of the cut edges along a segment of the degeneracy line for a mixed-type equation with a singular coefficient. The problem is investigated TF in the case $-m/2<\beta_0<1$ of , uniqueness of the solution of the problem TF and the existence of a solution TF to the singular Tricomi integral equation is proved.

Keywords: Regular operator, hypergeometric function, integral, Boltz formula, Wiener-Hopf equation, Unfredholm operators, Fourier transforms.

INTRODUCTION

The first fundamental research for a mixed-type equation was performed by the Italian mathematician Fr.Tricomi. After this work, the theory of problems for degenerate hyperbolic, elliptic, and mixed-type equations was developed in the fundamental research of foreign scientists E. Holmgren, S. Gellerstedt, and from domestic students, a significant contribution to the development of the theory of mixed-type equations was made in the works of M. S. Salakhitdinov, T. D. Dzhuraev, R. R. Ashurov, and B. Islomov.For further development of the theory of boundary value problems for mixed equations, an important place was occupied by the work (mixed problem) by A. M. Nakhushev, where in the hyperbolic part the nonlocal condition pointwise connects the values of the desired solution on both characteristics [1].

Problem statement TF

Consider the equation

$$(sign\,y)|y|^m u_{xx}+u_{yy}+(\beta_0/y)u_y=0, \qquad (1)$$

where $-(m/2)\le\beta_0<1$ is in a finite simply connected domain D of the complex plane $z=x+iy$ bounded by a normal curve $\sigma_0: x^2+4(m+2)^{-2}y^{m+2}=1$ with ends at points $A=A(-1,0)$, $B=B(1,0)$ and the characteristics AC and BC equations (1).

In the Tricomi AC problem, the value of the desired function is set at all points of the characteristic: $u(x,y)|_{AC}=\psi(x)$. In this chapter, we study the correctness of the problem where a part of the characteristic AC is freed from the boundary condition and this missing Tricomi condition is equivalently replaced by the non-local Frankl condition [2, 3] on different parts of the edges of the section along the degeneracy segment AB. Denote by D^+ and D^- the parts of the domain D that lie in the half –planes $y>0$ and $y<0$, respectively , and by C_0 and C_1, respectively, the points of intersection of the characteristics AC and BC with the characteristic originating from the point $E(c,0)$, where $c\in I=(-1,1)$ is the axis interval $y=0$. Let $p(x)\in C^1[-1,c]$ be a diffeomorphism from the set of points of the segment $[-1,c]$ to the set of points of the segment $[c,1]$, and $p'(x)<0$, $p(-1)=1$, $p(c)=c$. As an example of such a function, we give a linear function $p(x)=\delta-kx$, where $k=(1-c)/(1+c)$, $\delta=2c/(1+c)$.

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Problems for a mixed-type equation where a part of the characteristic AC is freed from the Tricomi boundary condition and this missing Tricomi condition is equivalently replaced by other conditions are investigated in [4,5,6,7].

A task TF **.** You need to find D a function in the scope $u(x,y)$ that meets the following conditions:

1. $u(x,y)$ - is continuous in each of the closed regions $\bar{D}^{+}$ and $\bar{D}^{-}$;

2. $u(x,y)\in C^{2}(D^{+})$ and satisfies equation (1) in this domain;

3. $u(x,y)$ is a generalized solution of the class R_1 [8,9,10] in the domain D^{-};

4. on the segment AB -line of the parabolic degeneracy of equation (2), the general gluing conditions [12] are satisfied

$$u(x,-0)=a(x)u(x,+0)=a_0(x),\ \ x\in\bar{I} \qquad (2)$$

$$\lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=b(x)\lim_{y\to +0}y^{\beta_0}\frac{\partial u}{\partial y}+b_0(x),\quad x\in I\setminus\{c\} \qquad (3)$$

where $a(x),b(x),a_0(x),b_0(x)$ are given continuously differentiable functions on $\bar{I}$, and $a(x)\neq 0, b(x)\neq 0,\ \ \forall x\in\bar{I},\ a_0(-1)=0,$ the limits (3) for $x=\pm 1$, $x=c$ may have singularities of order lower $1-2\beta$ than, where $\beta=(m+2\beta_0)/2(m+2)$.

5. Performed by

$$u(x,y)\big|_{\sigma_0}=\varphi(x)\ ,\qquad -1\le x\le 1; \qquad (4)$$

$$u(x,y)\big|_{AC_0}=\psi(x)\ ,\qquad -1\le x\le (c-1)/2; \qquad (5)$$

$$u(p(x),-0)=\mu(x)u(x,+0)+f(x)\ ,-1\le x\le c \qquad (6)$$

where $f(x)\in C[-1,c]\cap C^{1,\alpha}(-1,c),\quad f(-1)=0,\quad f(c)=0$, $\psi(x)\in C^{2}[-1,(c-1)/2]$, $\psi(-1)=0$ $\varphi(x)\in C^{0,\alpha}[-1,1]$, and

$$\varphi(x)=(1-x^{2})\tilde{\varphi}(x),\tilde{\varphi}(x)\in C^{0,\alpha}[-1,1]\ ,\ \ 0<\alpha<1. \qquad (7)$$

Condition (6) is analogousto the Frankl condition [2,3]. Linking the values of the desired function at the upper and lower edges of the sections along the segments $[-1,c]$ and, respectively $[c,1]$.

Let's introduce the notation

$$u(x,-0)=\tau^{-}(x),\ x\in\bar{I};\ \lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=\nu^{-}(x),\ x\in I \qquad (8)$$

$$u(x,+0)=\tau(x),\ x\in\bar{I};\ \lim_{y\to +0}y^{\beta_0}\frac{\partial u}{\partial y}=\nu(x),\ x\in I. \qquad (9)$$

By virtue of notations (8) and (9), the localFrankl displacement condition (6) takes the form

$$\tau(p(x))=\mu_1(x)\tau(x)+f_1(x),\ x\in(-1,c) \qquad (10)$$

where

$$\mu_1(x)=\mu(x)/a(p(x)),\ \ f_1(x)=(f(x)-a_0(p(x))/a(p(x))$$

and we write the general conjugation condition in the form

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$$\tau^{-}(x)=a(x)\tau(x)+a_0(x),\ \ x\in\bar{I} \qquad (11)$$

$$\nu^{-}(x)=b(x)\nu(x)+b_0(x),\ \ x\in I. \qquad (12)$$

Investigation of the problem TF in the case $-m/2<\beta_0<1$ and uniqueness of the solution of the problem TF.

By virtue of the Darboux formula D^{-}, the modified Cauchy problem with initial data (8), which gives a solution in the domain, is not difficult to obtain the following equation

$$\nu^{-}(x)=\gamma D_{-1,x}^{1-2\beta}\tau^{-}(x)+\Psi(x)\ \ ,\ \ \ x\in(-1,c)\ . \qquad (13)$$

Which is the first functional relation between $\tau(x)$ and $\nu(x)$ introduced on the axis $y=0$ from the region D^{-} where

$$\Psi(x)=-\gamma\left(\Gamma(\beta)/\Gamma(2\beta)\right)(1+x)^{\beta}D_{-1,x}^{1-\beta}\psi\left((x-1)/2\right),\ \ x\in(-1,c), \qquad (14)$$

$$\gamma=\frac{2\Gamma(2\beta)\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{m+2}{4}\right)^{2\beta}.$$

By virtue of the gluing conditions (11) and (12), relation (14) is written as

$$b(x)\nu(x)=\gamma\, D_{-1,x}^{1-2\beta}a(x)\tau(x)+\Psi_1(x),\ x\in(-1,c) \qquad (15)$$

$$\Psi_1(x)=\Psi(x)+\gamma\, D_{-1,x}^{1-2\beta}a_0(x)-b_0(x). \qquad (16)$$

Equality (14) is the first functional relation between unknown functions $\tau(x)$ and $\nu(x)$ the one introduced to the interval $(-1,c)$ from the domain D^{-}.

Theorem 1. *Solution of the problem TF for $\psi(x)\equiv 0,\ f(x)\equiv 0, a_0(x)\equiv b_0(x)\equiv 0$ and*

$$a'(x)\geq 0, a(x)>0, b(x)>0\ \ 0<\mu_1(x)<1 \qquad (17)$$

it reaches its positive maximum and negative minimum in a closed region $\bar{D}^{+}$ at the points of the curve σ_0.

Consequence. Problems TF in solving inequalities (17) have at most one solution.

Reducing the existence of a solution TF to a singular Tricomi integral equation.

Theorem 2. *Problem TF when the condition is met*

$$\lambda\pi\, k^{\frac{1}{2}-\alpha}\cos\alpha\pi<1 \qquad (18)$$

uniquely solvable, where $\lambda=\cos\beta\pi/\pi(1+\sin\beta\pi)$.

Proof. In the domain D^{+}, the solution of equation (1) satisfying the conditions

$$u(x,0)=\tau(x),\qquad x\in\bar{J}\ ;\qquad u\big|_{\sigma_0}=\varphi(x),\quad x\in\bar{J};$$

given by the formula

$$u(x,y)=k_2(1-\beta_0)y^{1-\beta_0}\int_{-1}^{1}\tau(t)\left\{\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}-\right.$$

$$\left.-\left[(1-xt)^2+\frac{4t^2}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right\}dt+k_2(1-\beta)(m+2)(1-R^2)y^{1-\beta_0}\times$$

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$$\times \int_{-1}^{1} \varphi(t)\left(r_1^2\right)^{\beta-2} F(1-\beta; 2-\beta, 2-2\beta; 1-\sigma)dt \quad , \qquad (19)$$

where

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = (x-t)^2 + \frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}} \mp \eta^{\frac{m+2}{2}}\right)^2 ,$$

$$R^2 = x^2 + \frac{4}{(m+2)^2} y^{m+2} , \quad \sigma = \frac{r^2}{r_1^2}, \quad (t,\eta) \in \sigma_0,$$

$$k_2 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta} \frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}.$$

We differentiate expression (19) by y :

$$\frac{\partial u}{\partial y} = k_2(1-\beta_0)\int_{-1}^{1} \tau(t)\frac{\partial}{\partial y} y^{1-\beta_0}\left\{\left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2}\right]^{\beta-1} - \right.$$

$$\left. -\left[(1-xt)^2 + \frac{4t^2}{(m+2)^2} y^{m+2}\right]^{\beta-1}\right\} dt + k_2(1-\beta)(m+2)\int_{-1}^{1} \varphi(t)\frac{\partial}{\partial y}\left\{(1-R^2)y^{1-\beta_o}\left(r_1^2\right)^{\beta-2} \times\right.$$

$$\left. \times F(1-\beta, 2-\beta, 2-2\beta; 1-\sigma)\right\} dt . \qquad (20)$$

It is not difficult to verify that

$$\frac{\partial}{\partial y}\left\{y^{1-\beta_o}\left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2}\right]^{\beta-1} - y^{1-\beta_0}\left[(1-xt)^2 + \frac{4}{(m+2)^2} y^{m+2}\right]^{\beta-1}\right\} =$$

$$= \frac{m+2}{2} y^{-\beta_0} \frac{\partial}{\partial t}\left\{(x-t)\left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2}\right]^{\beta-1} - \right.$$

$$\left. -\frac{1-xt}{x}\left[(1-xt)^2 + \frac{4}{(m+2)^2} y^{m+2}\right]^{\beta-1}\right\} . \qquad (21)$$

Now, in the first integral of the right-hand side of (20), taking into account identity (21), we perform the integration operation in parts, and, thus, we multiply the transformed relation (20) y^{β_0} by and then, passing to the limit $y \to 0$, we have [9]

$$v(x) = -k_2(1-\beta_0)\frac{m+2}{2}\left\{\frac{\tau(1)}{(1-x)^{1-2\beta}} + \frac{\tau(-1)}{(1+x)^{1-2\beta}} + \int_{-1}^{1} \frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\beta}} \right.$$

$$\left. -(2\beta-1)\int_{-1}^{1} \frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right\} + \Phi(x) \qquad (22)$$

where

$$\Phi(x) = k_2(1-\beta_0)(1-\beta)(m+2)(1-x^2)\int_{-1}^{1} \varphi(t)\left(x^2 - 2xt + 1\right)^{\beta-1} dt ,$$

here in force (7) $\tau(-1) = 0$, $\tau(1) = 0$.

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From functional relations (15) and (22) excluding $v(x)$, we have

$$\gamma D_{-1,x}^{1-2\beta} a(x)\tau(x) = -k_2(1-\beta_0)\frac{m+2}{2} b(x)\left\{\int_{-1}^{1}\frac{(x-t)\tau'(t)\,dt}{|x-t|^{2-2\beta}} - (2\beta-1)\int_{-1}^{1}\frac{\tau(t)\,dt}{(1-xt)^{2-2\beta}}\right\} + \Phi_0(x), \quad (23)$$

where

$$\Phi_0(x) = b(x)\Phi(x) - \Psi(x).$$

For convenience, in order to avoid cumbersome calculations [12], we further assume that $a(x) = b(x) = a = const.$ taking into account this assumption to equality (23), using the operator $D_{-1,x}^{2\beta-1}$, we obtain:

$$\gamma\tau(x) = -k_2(1-\beta_0)\frac{m+2}{2}\left\{D_{-1,x}^{2\beta-1}\left(\int_{-1}^{1}\frac{(x-t)\tau'(t)\,dt}{|x-t|^{2-2\beta}}\right) - (2\beta-1)\, D_{-1,x}^{2\beta-1}\left(\int_{-1}^{1}\frac{\tau(t)\,dt}{(1-xt)^{2-2\beta}}\right)\right\} + F_0(x), \quad (24)$$

where

$$F_0(x) = D_{-1,x}^{2\beta-1}\Phi_0(x)/a. \quad (25)$$

Converting the expression

$$D_{-1,x}^{2\beta-1}\left(\int_{-1}^{1}\frac{(x-t)\tau'(t)\,dt}{|x-t|^{2-2\beta}}\right) = \frac{1}{\Gamma(1-2\beta)}\int_{-1}^{x}\frac{ds}{(x-s)^{2\beta}}\int_{-1}^{1}\frac{(s-t)\tau'(t)\,dt}{|s-t|^{2-2\beta}}. \quad (26)$$

Calculate the internal integral

$$I_1 = \int_{-1}^{1}\frac{(s-t)\tau'(t)\,dt}{|s-t|^{2-2\beta}} = \lim_{\varepsilon\to 0}\left(\int_{-1}^{s-\varepsilon}\frac{\tau'(t)\,dt}{(s-t)^{1-2\beta}} - \int_{s+\varepsilon}^{1}\frac{\tau'(t)\,dt}{(t-s)^{1-2\beta}}\right).$$

Here we perform the integration operation in parts :

$$I_1 = \lim_{e\to 0}\left(\tau(s-\varepsilon)\varepsilon^{2\beta-1} - \tau(-1)(1+s)^{2\beta-1} + (2\beta-1)\int_{-1}^{s-\varepsilon}\frac{\tau(t)\,dt}{(s-t)^{2-2\beta}} - \right.$$

$$\left. -\tau(1)(1-s)^{2\beta-1} + \tau(s+\varepsilon)\varepsilon^{2\beta-1} + (2\beta-1)\int_{s+\varepsilon}^{1}\frac{\tau(t)\,dt}{(t-s)^{2-2\beta}}\right).$$

Taking into account the equalities

$$(2\beta-1)\int_{-1}^{s-\varepsilon}\frac{\tau(t)\,dt}{(s-t)^{2-2\beta}} = \frac{d}{ds}\int_{-1}^{s-\varepsilon}\frac{\tau(t)\,dt}{(s-t)^{1-2\beta}} - \tau(s-\varepsilon)\varepsilon^{2\beta-1},$$

$$(2\beta-1)\int_{s+\varepsilon}^{1}\frac{\tau(t)\,dt}{(t-s)^{2-2\beta}} = -\frac{d}{ds}\int_{s+\varepsilon}^{1}\frac{\tau(t)\,dt}{(t-s)^{1-2\beta}} - \tau(s+\varepsilon)\varepsilon^{2\beta-1}$$

in I_1 passing to the limit at $\varepsilon \to 0$, we obtain

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$$I_1=\int_{-1}^{1}\frac{(s-t)\tau'(t)dt}{|s-t|^{2-2\beta}}=-\tau(-1)(1+s)^{2\beta-1}+\frac{d}{ds}\int_{-1}^{s}\frac{\tau(t)dt}{(s-t)^{1-2\beta}}-$$

$$-\tau(1)(1-s)^{2\beta-1}-\frac{d}{ds}\int_{s}^{1}\frac{\tau(t)dt}{(t-s)^{1-2\beta}}. \qquad (27)$$

Thus, by virtue of (27), relation (26) has the form

$$D^{2\beta-1}_{-1,x}\left(\int_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|x-t|^{2-2\beta}}\right)= \Gamma(2\beta)\ D^{2\beta-1}_{-1,x}D^{1-2\beta}_{-1x}\ \tau(x)+\Gamma(2\beta)\ D^{2\beta-1}_{-1,x}\ D^{1-2\beta}_{x,1}\tau(x). \qquad (28)$$

Using the formula

$$D^{-\infty}_{a,x}D^{\alpha}_{x,b}\Phi(t)=\cos\alpha\pi\Phi(x)-\frac{\sin\alpha\pi}{\pi}\int_{a}^{b}\left(\frac{x-a}{t-a}\right)^{\alpha}\frac{\Phi(t)dt}{t-x},$$

relation (28) is rewritten as

$$D^{2\beta-1}_{-1,x}\left(\int_{-1}^{1}\frac{(x-t)\tau'(t)dt}{|\ x-t\ |^{2-2\beta}}\right)= \Gamma(2\beta)\tau(x)+\Gamma(2\beta)cos(1-2\beta)\pi\tau(x)-$$

$$-\frac{1}{\tilde{A}(1-2\beta)}\int_{-1}^{1}\left(\frac{1+\tilde{o}}{1+t}\right)^{1-2\beta}\frac{\tau(t)dt}{t-x}. \qquad (29)$$

It is also not hard to get that

$$D^{2\beta-1}_{-1,x}\left(\int_{-1}^{1}\frac{\tau(t)dt}{(1-xt)^{2-2\beta}}\right)=\frac{1}{\tilde{A}(2-2\beta)}\int_{-1}^{1}\left(\frac{1+\tilde{o}}{1+t}\right)^{1-2\beta}\frac{\tau(t)dt}{1-xt} \qquad (30)$$

Now, substituting (29) - (30) into (24), we obtain the following singular integral equation with respect to the unknown function $\tau(x)$:

$$\tau(x)-\lambda\int_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt=F(x), \qquad (31)$$

where $\lambda=\cos\beta\pi/\pi(1+\sin\beta\pi)$,

$F(x)=F_0(x)/\gamma$.

Note that the kernel of equation (31) has a singular singularity only for $t\in(-1,c)$, since, equation (31) holds only for $x\in(-1,c)$, taking this remark into account, we transform equations (31) to the form

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt-$$

$$-\lambda\int_{c}^{1}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt=F(x),\quad x\in(-1,c). \qquad (32)$$

In the interval integral $(c,1)$, the right-hand side of (32), by replacing the variable integration $t=p(s)=\delta-ks$, taking into account equality $\tau(p(s))=\tau(s)+f(s)$, we obtain

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt+$$

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$$+\lambda\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}\left(\frac{1}{p(s)-x}-\frac{1}{1-xp(s)}\right)(\tau(s)+f(s))p'(s)ds=F(x)\quad,\ -1\le x\le c \quad .\ (33)$$

Equations (33) are transformed to the form

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt=-\lambda\int_{-1}^{c}\frac{p'(s)\tau(s)}{p(s)-x}ds+R[\tau]+F_1(x),\quad (34)$$

where

$$R[\tau]=-\lambda\int_{-1}^{c}\left[\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}-1\right]\frac{p'(s)\tau(s)ds}{p(s)-x}+\lambda\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}\frac{p'(s)\tau(s)}{1-xp(s)}ds \quad (35)$$

-regular operator,

$$F_1(x)=F(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-2\beta}\left(\frac{1}{p(s)-x}-\frac{1}{1-xp(s)}\right)p'(s)f(s)ds \quad .$$

The first integral operator of the right-hand side of (34) is not regular, since the integral expression under $x=c,\ \ s=c$ has an isolated first-order singularity, and therefore this term in (34) is distinguished separately.

Temporarily assuming that the right-hand side of equation (34) is a known function, we rewrite it as

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\tau(t)dt=g_0(x)\quad,\ \ x\in(-1,c),\quad (35a)$$

where

$$g_0(x)=-\lambda\int_{-1}^{c}\frac{p'(s)\tau(s)ds}{p(s)-x}+R[\tau]+F_1(x) \quad . \quad (36)$$

Theorem 2 is proved.

Regularization of the singular Tricomi integral equation

In equation (35a), we introduce the notation $\rho(x)=(1+x)^{2\beta-1}\tau(x)$, $g(x)=(1+x)^{2\beta-1}g_0(x)$, and transform it to the form

$$\rho(x)-\lambda\int_{-1}^{c}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt=g(x)\quad,\ -1\le x\le c \quad . \quad (37)$$

We will look for the solution of the singular integral equation (37) in the class of functions bounded at a point $x=c$ and admitting a singularity of order lower $1-2\beta$ at a point $x=-1$, i.e. in the class $h(c)$.

To regularize the singular integral equation (37), we apply the Carleman-Vekua method developedby S. G. Mikhlin [13].

Let z be an arbitrary point in the complex plane z. In this plane, we introduce the function

$$\Phi(z)=\frac{1}{2\pi i}\int_{-1}^{c}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)\rho(t)dt \quad (38)$$

By Δ denote the following infinite intervalsof the real axis

$$\Delta=\begin{cases}\left(\frac{1}{c},-1\right), & \text{если } c<0,\\ (-\infty,-1), & \text{если } c=0,\\ (-\infty,-1)\cup\left(\frac{1}{c},+\infty\right), & \text{если } c>0.\end{cases}$$

The function $\Phi(z)$ is holomorphic in the entire plane z except for the points of the set $(-1,c)\cup\Delta$ of the real axis, and $\Phi(z)\to 0$ if $\operatorname{Im} z\to\infty$.

By virtue of the Sokhotsky-Plemel formulas on the interval $-1<x<c$, we have

$$\Phi^+(x)-\Phi^-(x)=\rho(x), \qquad (39)$$

$$\Phi^+(x)+\Phi^-(x)=\frac{1}{\pi i}\int_{-1}^{c}\left(\frac{1}{t-x}-\frac{1}{1-xt}\right)\rho(t)dt. \qquad (40)$$

Here $\Phi^+(x)$ are the $\Phi^-(x)$ limit values of the function $\Phi(z)$, when the point z tends to the real axis, respectively, from the upper or lower half-plane.

By virtue of formulas (39) and (40), equation (37) is rewritten as

$$(1-\lambda\pi i)\Phi^+(x)-(1+\lambda\pi i)\Phi^-(x)=g(x), \quad -1<x<c. \qquad (41)$$

It is easy to see that the relation holds for the function $\Phi(z)$ имеет место соотношение

$$\Phi\left(\frac{1}{z}\right)=z\Phi(z). \qquad (42)$$

The transformation $W=\frac{1}{z}$ maps the upper half-plane to the lower half-plane and vice versa. In this case, the interval $(-1,c)$ is displayed in infinite intervals Δ.

In (41) x, replacing by $\frac{1}{x}$, taking into account the relations

$$\Phi^+\left(\frac{1}{x}\right)=x\Phi^-(x), \quad \Phi^-\left(\frac{1}{x}\right)=x\Phi^+(x) \qquad (43)$$

we get

$$(1+\lambda\pi i)\Phi^+(x)-(1-\lambda\pi i)\Phi^-(x)=-\frac{1}{x}g\left(\frac{1}{x}\right), \qquad (44)$$

here $1+\lambda\pi i=\frac{e^{\alpha\pi i}}{\cos\alpha\pi}, \quad 1-\lambda\pi i=\frac{e^{-\alpha\pi i}}{\cos\alpha\pi}, \quad \alpha=(1-2\beta)/4$

Let's introduce the following functions

$$G(x)=\begin{cases}\frac{1+\lambda\pi i}{1-\lambda\pi i}=e^{2\alpha\pi i}, & \text{если} \quad -1<x<c,\\ \frac{1-\lambda\pi i}{1+\lambda\pi i}=e^{-2\alpha\pi i}, & \text{если} \quad x\in\Delta,\end{cases} \qquad (45)$$

$$h(x)=\begin{cases}\cos\alpha\pi e^{\alpha\pi i}g(x). & \text{если} \quad -1<x<c,\\ -\cos\alpha\pi e^{-\alpha\pi i}\frac{1}{x}g\left(\frac{1}{x}\right), & \text{если} \quad x\in\Delta.\end{cases} \qquad (46)$$

Using these functions, equations (41) and (44) can be combined into one equation

$$\Phi^{+}(x)-G(x)\Phi^{-}(x)=h(x) \quad , \quad x\in I\cup\Delta . \qquad (47)$$

Thus, the problem of finding a solution to the singular integral equation (46) was reduced to the Riemann problem of the theory of functions of a complex variable: find a function that vanishes at infinity $\Phi(z)$, is holomorphic, both in the upper and lower half-plane, and on the real axis satisfies condition (46).

First, we solve the following homogeneous problem: find a function bounded at infinity $X(z)$ that is holomorphic in both the upper half-plane and the lower half-plane and $y=0$ satisfies the condition on the real axis

$$X^{+}(x)-G(x)X^{-}(x)=0 \quad , \quad x\in(-1,c)\cup\Delta ,$$

or

$$\ln X^{+}(x)-\ln X^{-}(x)=\ln G(x) . \qquad (48)$$

One of the partial solutions to this problem has the form

$$\ln X(z)=\frac{1}{2\pi i}\int_{-1}^{c}\left(\frac{1}{t-z}-\frac{z}{1-zt}\right)\ln G(t)\,dt , \qquad (49)$$

or

$$X(z)=\exp\{\alpha[\ln(c-z)+\ln(1-cz)-\ln(-1-z)-\ln(1+z)]\} .$$

From here, it's not hard to see that

$$X^{+}(x)=R(x)e^{\alpha\pi i},\ X^{-}(x)=R(x)e^{-\alpha\pi i} \ , \ x\in(-1,c),$$

$$X^{+}(x)=R(x)e^{-\alpha\pi i},\ X^{-}(x)=R(x)e^{\alpha\pi i} \ , \ x\in\Delta, \qquad (50)$$

where

$$R(x)=\left(\frac{(c-x)(1-cx)}{(1+x)^{2}}\right)^{\alpha} . \qquad (51)$$

Taking into account equality (50) and (51), we rewrite equation (47) as

$$\frac{\Phi^{+}(x)}{X^{+}(x)}-\frac{\Phi^{-}(x)}{X^{-}(x)}=\frac{h(x)}{X^{+}(x)} \quad , \quad x\in(-1,c)\cup\Delta . \qquad (52)$$

One of the partial solutions of this equation has the form

$$\frac{\Phi(z)}{X(z)}=\frac{1}{2\pi i}\left[\int_{-1}^{c}\frac{h(t)dt}{X^{+}(t-z)}+\int_{\Delta}\frac{h(t)dt}{X^{+}(t-z)}\right]. \qquad (53)$$

In the second integral of the right-hand side of (53), by substituting переменthe integration variablesho интегрирования $t=\frac{1}{\xi}$ o and performing the necessary calculations taking into account (46) , andh equalities $X^{+}\left(\frac{1}{\xi}\right)=X^{-}(\xi) \quad , \quad X^{-}(\xi)=X^{+}(\xi)e^{-2\alpha\pi i}$, it is not difficult to verify that

$$\frac{\Phi(z)}{X(z)}=\frac{1}{2\pi i}\int_{-1}^{c}\frac{\cos\alpha\pi\, e^{\alpha\pi i}g(t)}{X^{+}(t)}\left(\frac{1}{t-z}-\frac{1}{1-zt}\right)dt . \qquad (54)$$

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